

QUANTUM CODES THROUGH CONSTACYCLIC CODES OVER
 $\mathcal{F}_p + \mu\mathcal{F}_p + \nu\mathcal{F}_p + \varpi\mathcal{F}_p + \mu\nu\mathcal{F}_p + \nu\varpi\mathcal{F}_p + \mu\varpi\mathcal{F}_p + \mu\nu\varpi\mathcal{F}_p$

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Abstract: The purpose of this paper is to identify the structural characteristics and construction of quantum codes over \mathcal{F}_p using $\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi$ -constacyclic codes over ring $\mathcal{R} = \mathcal{F}_p[\mu, \nu, \varpi] / \langle \mu^2 - \mu, \nu^2 - \nu, \varpi^2 - \varpi, \mu\nu - \nu\mu, \nu\varpi - \varpi\nu, \mu\varpi - \varpi\mu \rangle$, here \mathcal{F}_p is a finite field with p elements. We define a Gray map from \mathcal{R} to \mathcal{F}_p^8 , a distance-preserving map. Breaking down constacyclic codes into cyclic and negacyclic codes results in the creation of quantum codes over the finite field \mathcal{F}_p . As an application, some examples are illustrated to obtain the quantum codes of different parameters.

Keywords and Phrases: Gray map, cyclic codes, constacyclic codes, negacyclic codes.

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1. Introduction

Quantum error correcting(QEC) codes are extremely useful for quantum computation as well as quantum communication. An effective method for overcoming decoherence is provided by QEC codes. As the QEC code was first discovered by Shor [15], in subsequent research, Calderbank et al. [7] developed a measure of distinction between QEC codes and conventional error-correcting codes.

Later in [8], Gao constructed the quantum codes through cyclic codes over $F_q + vF_q + v^2F_q + v^3F_q$. Ashraf and Mohammad [2] constructed the constacyclic codes

over $F_p[u, v]/\langle u^2 - 1, v^3 - v, uv - vu \rangle$. Some authors have formulated quantum codes by overlaying the Gray image of cyclic codes onto specific finite rings.

Ashraf et al. [1, 3] made notable progress in constructing quantum codes by utilizing cyclic codes. Several researchers have developed numerous new quantum codes by leveraging classical cyclic and constacyclic codes [5, 10, 11, 12, 13, 14, 16, 17]. Bag et al. [6] established the applications of constacyclic codes over the semi local ring $F_{p^m} + vF_{p^m}$. Alkenani et al. [4] worked on the problem of the quantum codes from constacyclic codes over $F[u_1, u_2]/\langle u_1^2 - u_1, u_2^2 - u_2, u_1u_2 - u_2u_1 \rangle$. These studies suggest that exploration of generalized ring structures and their corresponding constacyclic codes may lead to new families of quantum codes possessing improved parameters. Gowdhaman et al. [9] analyzed constacyclic codes over the non-chain finite commutative ring $\mathbb{Z}_4[u, v]/\langle u^2 - v, v^2, uv \rangle$. Yadav et al. [18] constructed quantum codes via the CSS method using cyclic codes over the commutative non-local ring $F_p \times (F_p + \alpha F_p + \alpha^2 F_p)$. Motivated by these developments, considerable attention has been devoted to constructing quantum codes from well-structured classical codes defined over finite fields and rings. We generalize this structure $F_p[u_1, u_2]/\langle u_1^2 - u_1, u_2^2 - u_2, u_1u_2 - u_2u_1 \rangle$ and construct various parameters of new quantum codes through constacyclic codes.

Encouraged by such problems, the present work focusses on the study of quantum codes through constacyclic codes over $\mathcal{F}_p[\mu, \nu, \varpi]/\langle \mu^2 - \mu, \nu^2 - \nu, \varpi^2 - \varpi, \mu\nu - \nu\mu, \nu\varpi - \varpi\nu, \mu\varpi - \varpi\mu \rangle$. In section 2, we describe the general form of arbitrary elements of the ring $\mathcal{R} = \mathcal{F}_p[\mu, \nu, \varpi]/\langle \mu^2 - \mu, \nu^2 - \nu, \varpi^2 - \varpi, \mu\nu - \nu\mu, \nu\varpi - \varpi\nu, \mu\varpi - \varpi\mu \rangle$ and introduce the auxiliary definitions required for subsequent developments. In section 3, we define the Gray map over \mathcal{R} . Finally, in section 4, we construct quantum codes through constacyclic codes over this ring. In section 5, we present a set of results that serve as valuable tools for determining the parameters of quantum codes and we conclude our work in section 6.

2. Preliminaries

Throughout this paper, \mathcal{F}_p represents a finite field with p elements, where p is an odd prime power. Let $\mathcal{R} = \mathcal{F}_p + \mu\mathcal{F}_p + \nu\mathcal{F}_p + \varpi\mathcal{F}_p + \mu\nu\mathcal{F}_p + \nu\varpi\mathcal{F}_p + \mu\varpi\mathcal{F}_p + \mu\nu\varpi\mathcal{F}_p$ be the ring such that $\mu^2 = \mu, \nu^2 = \nu, \varpi^2 = \varpi, \mu\nu = \nu\mu, \nu\varpi = \varpi\nu, \mu\varpi = \varpi\mu$.

Definition 1. A λ -constacyclic code is defined as a linear code C , where every codeword $(d_0, d_1, \dots, d_{n-1}) \in C$ satisfies $(\lambda d_{n-1}, d_0, d_1, \dots, d_{n-2}) \in C$.

If $\lambda = 1$, then codes becomes **cyclic**.

If $\lambda = -1$, then codes becomes **negacyclic**.

Definition 2. The codeword $d = (d_0, d_1, \dots, d_{n-1}) \in \mathcal{R}^n$ can be identified in the form of a polynomial $d(x) = d_0 + d_1x + \dots + d_{n-1}x^{n-1}$ in $\mathcal{R}[x]/\langle x^n - \lambda \rangle$.

Definition 3. The Euclidean inner product of two elements r and h is defined as $r.h = \sum_{i=0}^{n-1} r_i h_i$, where $r = (r_0, r_1, \dots, r_{n-1})$ and $h = (h_0, h_1, \dots, h_{n-1})$. If their inner product is zero then the elements are orthogonal.

Definition 4. The dual code for a code C is as follows:
 $C^\perp = \{r \in \mathcal{R}^n : r.h = 0 \ \forall h \in C\}$. A code C is referred to as self-orthogonal or self-dual, which depends on whether $C \subseteq C^\perp$ or $C = C^\perp$ respectively.

Definition 5. Lee weight of any codeword r is written as $w_L(r) = w_H(\psi(r))$, where $w_H(\psi(r))$ represents the Hamming weight. It is the number of non-zero components in a codeword that determines its Hamming weight.

Definition 6. Let r_1 and r_2 be any two codewords. Then the Lee distance is $d_L(r_1, r_2) = w_L(r_1 - r_2) = w_H(\psi(r_1 - r_2))$.
 Minimum distance of C is as $d_L(C) = \min\{d_L(r_1, r_2) | r_1 \neq r_2\}$.

The elements used in \mathcal{R} can be written in the following manner:

$$\begin{aligned} & \alpha_1 + \alpha_2 u + \alpha_3 v + \alpha_4 \varpi + \alpha_5 \mu\nu + \alpha_6 \nu\varpi + \alpha_7 \mu\varpi + \alpha_8 \mu\nu\varpi \\ & = \alpha_1(1 - \mu - \nu - \varpi + \mu\nu + \nu\varpi + \mu\varpi + \mu\nu\varpi) \\ & + (\alpha_1 + \alpha_2)(\mu - \mu\nu - \mu\varpi + \mu\nu\varpi) + (\alpha_1 + \alpha_3)(\nu - \mu\nu - \nu\varpi + \mu\nu\varpi) \\ & + (\alpha_1 + \alpha_4)(\varpi - \mu\varpi - \nu\varpi + \mu\nu\varpi) + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5)(\mu\nu - \mu\nu\varpi) \\ & + (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_6)(\nu\varpi - \mu\nu\varpi) + (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_7)(\mu\varpi - \mu\nu\varpi) \\ & + (\sum_{i=1}^8 \alpha_i) \mu\nu\varpi = \sum_{i=1}^8 \dot{\alpha}_i \varphi_i, \quad \dots(*) \end{aligned}$$

where $\dot{\alpha}_1 = \alpha_1, \dot{\alpha}_s = \alpha_1 + \alpha_s$, where $s = 2, 3, 4$,
 $\dot{\alpha}_5 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \dot{\alpha}_6 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_6, \dot{\alpha}_7 = \alpha_1 + \alpha_2 + \alpha_4 + \alpha_7, \dot{\alpha}_8 = \sum_{i=1}^8 \alpha_i$,
 $\varphi_1 = 1 - \mu - \nu - \varpi + \mu\nu + \nu\varpi + \mu\varpi + \mu\nu\varpi, \varphi_2 = \mu - \mu\nu - \mu\varpi + \mu\nu\varpi, \varphi_3 = \nu - \mu\nu - \nu\varpi + \mu\nu\varpi, \varphi_4 = \varpi - \mu\varpi - \nu\varpi + \mu\nu\varpi, \varphi_5 = \mu\nu - \mu\nu\varpi, \varphi_6 = \nu\varpi - \mu\nu\varpi, \varphi_7 = \mu\varpi - \mu\nu\varpi$ and $\varphi_8 = \mu\nu\varpi$.

Therefore, we have

- (i) For any $i \neq j$, we have $\varphi_i^2 = \varphi_i$ and $\varphi_i \varphi_j = 0$, where $i, j = 1, 2, \dots, 8$.
- (ii) $\sum_{i=1}^8 \varphi_i = 1$, where $i \in [1, 8]_{\mathbb{Z}}$.

By using Chinese Remainder theorem, the ring \mathcal{R} can be expressed as $\mathcal{R} = \bigoplus_{i=1}^8 \varphi_i \mathcal{R} = \bigoplus_{i=1}^8 \varphi_i \mathcal{F}_p$. Moreover, it can be observed that $\varphi_i \mathcal{R} \cong \mathcal{F}_p$ for $1 \leq i \leq 8$. Consequently, it is possible to express any $r \in \mathcal{R}$ in a unique way as $r = \sum_{i=1}^8 \varphi_i \dot{a}_i$, where $\dot{a}_i \in \mathcal{F}_p$ and $i=1, 2, \dots, 8$.

3. Gray Map over \mathcal{R}

Gray map is defined as follows:

$$\begin{aligned} & \psi : \mathcal{R} \rightarrow \mathcal{F}_p^8, \\ & \psi(r) = (\dot{a}, \dot{b}, \dot{c}, \dot{d}, \dot{e}, \dot{f}, \dot{g}, \dot{h}), \text{ where } \dot{a}, \dot{b}, \dot{c}, \dot{d}, \dot{e}, \dot{f}, \dot{g}, \dot{h} \in \mathcal{F}_p. \end{aligned}$$

This map can be naturally

extended to \mathcal{R}^n

$$\psi : \mathcal{R}^n \rightarrow \mathcal{F}_p^{8n}$$

$r = (\mathbf{r}_0, \mathbf{r}_1, \dots, \mathbf{r}_{n-1}) \mapsto (\dot{a}_0, \dot{a}_1, \dots, \dot{a}_{n-1}, \dot{b}_0, \dot{b}_1, \dots, \dot{b}_{n-1}, \dot{c}_0, \dot{c}_1, \dots, \dot{c}_{n-1}, \dot{d}_0, \dot{d}_1, \dots, \dot{d}_{n-1}, \dot{e}_0, \dot{e}_1, \dots, \dot{e}_{n-1}, \dot{f}_0, \dot{f}_1, \dots, \dot{f}_{n-1}, \dot{g}_0, \dot{g}_1, \dots, \dot{g}_{n-1}, \dot{h}_0, \dot{h}_1, \dots, \dot{h}_{n-1})$,
 where $\mathbf{r}_j = \wp_1 \dot{a}_j + \wp_2 \dot{b}_j + \wp_3 \dot{c}_j + \wp_4 \dot{d}_j + \wp_5 \dot{e}_j + \wp_6 \dot{f}_j + \wp_7 \dot{g}_j + \wp_8 \dot{h}_j$ and $\dot{a}_j, \dot{b}_j, \dot{c}_j, \dot{d}_j, \dot{e}_j, \dot{f}_j, \dot{g}_j, \dot{h}_j \in \mathcal{F}_p$ for $j \in [0, n-1]_{\mathbb{Z}}$.

Theorem 1. A Gray map is a linear map that preserves distance from \mathcal{R}^n to \mathcal{F}_p^{8n} .

Proof. Let $\dot{r} = \sum_{i=1}^8 \dot{a}_i \wp_i$ and $\ddot{r} = \sum_{i=1}^8 \ddot{a}_i \wp_i$, where $\dot{a}_i, \ddot{a}_i \in \mathcal{F}_p^n$ for $1 \leq i \leq 8$,
 then $\psi(\dot{r} + \ddot{r}) = (\dot{a}_1 + \ddot{a}_1, \dot{a}_2 + \ddot{a}_2, \dot{a}_3 + \ddot{a}_3, \dot{a}_4 + \ddot{a}_4, \dot{a}_5 + \ddot{a}_5, \dot{a}_6 + \ddot{a}_6, \dot{a}_7 + \ddot{a}_7, \dot{a}_8 + \ddot{a}_8)$
 $= (\dot{a}_1, \dot{a}_2, \dot{a}_3, \dot{a}_4, \dot{a}_5, \dot{a}_6, \dot{a}_7, \dot{a}_8) + (\ddot{a}_1, \ddot{a}_2, \ddot{a}_3, \ddot{a}_4, \ddot{a}_5, \ddot{a}_6, \ddot{a}_7, \ddot{a}_8)$
 $= \psi(\dot{r}) + \psi(\ddot{r})$

Let $\dot{r} = \sum_{i=1}^8 \dot{a}_i \wp_i \in \mathcal{R}^n$ and $\beta \in \mathcal{F}_p$ then $\psi(\beta \dot{r}) = \psi(\sum_{i=1}^8 \beta \dot{a}_i \wp_i)$
 $= \beta \psi(\dot{r})$

So ψ is \mathcal{F}_p -linear and $\mathfrak{d}_L(\dot{r}, \ddot{r}) = \mathfrak{w}_L(\dot{r} - \ddot{r}) = \mathfrak{w}_H(\psi(\dot{r} - \ddot{r}))$

$= \mathfrak{w}_H((\psi(\dot{r}) - (\psi(\ddot{r}))) = \mathfrak{d}_H((\psi(\dot{r}), (\psi(\ddot{r})))$

Hence, the Gray map ψ preserves distances.

Theorem 2. If C is a linear code over \mathcal{R} of length n with $|C| = p^k$ and $d_L(C) = \mathfrak{d}_L$ then $\psi(C)$ is a p -ary linear code having $[8n, k, \mathfrak{d}_H]$.

Proof. By theorem 1, we have ψ is bijective map. So $|C| = |\psi(C)| = p^k$ and $\psi(C)$ has length $8n$, it follows that $\psi(C)$ represents an $[8n, k, \mathfrak{d}_H]$ linear code over \mathcal{F}_p^{8n} , where $\mathfrak{d}_L = \mathfrak{d}_H$.

Theorem 3. If C is self-orthogonal, then $\psi(C)$ is self-orthogonal and hence $\psi(C^\perp) = \psi(C)^\perp$.

Proof. Let us consider $r_1 = \sum_{k=1}^8 \dot{a}_k \wp_k \in C$ and $r_2 = \sum_{k=1}^8 \dot{b}_k \wp_k \in C$. Since $r_1 r_2 = \sum_{k=1}^8 \wp_k \dot{a}_k \dot{b}_k = 0$ which implies $\dot{a}_k \dot{b}_k = 0$, where $k \in [1, 8]_{\mathbb{Z}}$.

Also, $\psi(r_1) \cdot \psi(r_2) = (\dot{a}_1, \dot{a}_2, \dot{a}_3, \dot{a}_4, \dot{a}_5, \dot{a}_6, \dot{a}_7, \dot{a}_8) \cdot (\dot{b}_1, \dot{b}_2, \dot{b}_3, \dot{b}_4, \dot{b}_5, \dot{b}_6, \dot{b}_7, \dot{b}_8) = \dot{a}_1 \dot{b}_1 + \dot{a}_2 \dot{b}_2 + \dots + \dot{a}_8 \dot{b}_8 = 0$.

Since $\psi(C)$ is self-orthogonal therefore $\psi(r_2) \in \psi(C)^\perp$, as $\psi(r_1) \in \psi(C)$. Thus, we have $\psi(C^\perp) \subseteq \psi(C)^\perp$. Also ψ is bijection, so $|\psi(C^\perp)| = |\psi(C)^\perp|$. Hence, the result is proved.

Theorem 4. If C is a linear code, then $\psi(C) = \otimes_{k=1}^8 C_k$.

Proof. Since $C = \oplus_{k=1}^8 \wp_k C_k$ and each $s \in C$ can be written as $s = \sum_{k=1}^8 \wp_k \dot{a}_k$, where $\dot{a}_k \in C_k$ and $k \in [1, 8]_{\mathbb{Z}}$. With the help of the Gray map, it can be easily seen that $\psi(s) = (\dot{a}, \dot{b}, \dot{c}, \dot{d}, \dot{e}, \dot{f}, \dot{g}, \dot{h}) \in \otimes_{k=1}^8 C_k$. Hence $\psi(C) = \otimes_{k=1}^8 C_k$, as ψ is a bijective map.

Proposition 5. *If C is a linear code over \mathcal{R} , then $C^\perp = \bigoplus_{i=1}^8 \wp_i C_i^\perp$ is also a linear code over \mathcal{R} .*

4. Quantum Codes from Constacyclic Codes

Now, $\forall i \in [1, 8]_{\mathbb{Z}}$, we define a linear code $C_i \subseteq \mathcal{F}_p^n$
 $C_i = \{a_i \in \mathcal{F}_p^n \mid \exists a_j \in \mathcal{F}_p^n, \text{ where } i \neq j \text{ and } i, j \in [1, 8]_{\mathbb{Z}} \text{ such that } \sum_{i=1}^8 \wp_i a_i \in C\}$.
 For $i=1,2,\dots,8$, C_i are linear codes over \mathcal{F}_p^n then C_i are p-ary linear codes of length n . Additionally, a linear code C can be uniquely expressed in the following manner
 $C = \bigoplus_{i=1}^8 \wp_i C_i$ (**)
 & $|C| = \prod_{i=1}^8 |C_i|$.
 Let G denote the matrix generator of a linear code C over \mathcal{R} . Then from (**), we get

$$G = \begin{pmatrix} \wp_1 G_1 \\ \wp_2 G_2 \\ \wp_3 G_3 \\ \dots \\ \wp_8 G_8 \end{pmatrix},$$

where for all $i \in [1, 8]_{\mathbb{Z}}$, G_i are the generator matrices of C_i and the generator matrix of $\psi(C)$ is defined in the following manner

$$\psi(G) = \begin{pmatrix} \psi(\wp_1 G_1) \\ \psi(\wp_2 G_2) \\ \psi(\wp_3 G_3) \\ \dots \\ \psi(\wp_8 G_8) \end{pmatrix}$$

Theorem 6. *If $(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)$ is a unit in \mathcal{R} and $C = \bigoplus_{i=1}^8 \wp_i C_i$ then C is a $(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)$ -constacyclic codes of length n over \mathcal{R} iff C_i 's are α_i -constacyclic code over \mathcal{F}_p respectively, where $i \in [1, 8]_{\mathbb{Z}}$.*

Proof. Let C be a $(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)$ -constacyclic code over \mathcal{R} and

$$\begin{aligned} \dot{a} &= (\dot{a}_0, \dot{a}_1, \dots, \dot{a}_{n-1}) \in C_1, & \dot{b} &= (\dot{b}_0, \dot{b}_1, \dots, \dot{b}_{n-1}) \in C_2, \\ \dot{c} &= (\dot{c}_0, \dot{c}_1, \dots, \dot{c}_{n-1}) \in C_3, & \dot{d} &= (\dot{d}_0, \dot{d}_1, \dots, \dot{d}_{n-1}) \in C_4, \\ \dot{e} &= (\dot{e}_0, \dot{e}_1, \dots, \dot{e}_{n-1}) \in C_5, & \dot{f} &= (\dot{f}_0, \dot{f}_1, \dots, \dot{f}_{n-1}) \in C_6, \\ \dot{g} &= (\dot{g}_0, \dot{g}_1, \dots, \dot{g}_{n-1}) \in C_7, & \dot{h} &= (\dot{h}_0, \dot{h}_1, \dots, \dot{h}_{n-1}) \in C_8, \end{aligned}$$

where $\dot{a}_i, \dot{b}_i, \dot{c}_i, \dot{d}_i, \dot{e}_i, \dot{f}_i, \dot{g}_i, \dot{h}_i \in \mathcal{F}_p$ for $i=0,1,2,\dots,n-1$.

Let $\gamma = (\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_{n-1}) \in C$,

where $\gamma_i = \wp_1 \dot{a}_i + \wp_2 \dot{b}_i + \wp_3 \dot{c}_i + \wp_4 \dot{d}_i + \wp_5 \dot{e}_i + \wp_6 \dot{f}_i + \wp_7 \dot{g}_i + \wp_8 \dot{h}_i$. (***)

Since C is a $(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)$ -constacyclic codes of length n over \mathcal{R} . So

$$\Upsilon_{(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)} \\ = ((\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)\gamma_{n-1}, \gamma_0, \dots, \gamma_{n-2}).$$

By using (*) and (***), we have

$$\Upsilon_{(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)} \\ = (\wp_1 \dot{a}_{n-1} \dot{a}_1 + \wp_2 \dot{b}_{n-1} \dot{a}_2 + \wp_3 \dot{c}_{n-1} \dot{a}_3 + \wp_4 \dot{d}_{n-1} \dot{a}_4 + \wp_5 \dot{e}_{n-1} \dot{a}_5 + \wp_6 \dot{f}_{n-1} \dot{a}_6 \\ + \wp_7 \dot{g}_{n-1} \dot{a}_7 + \wp_8 \dot{h}_{n-1} \dot{a}_8, \gamma_0, \gamma_1, \dots, \gamma_{n-2}) \\ = \wp_1(\dot{a}_1 \dot{a}_{n-1}, \dot{a}_0, \dot{a}_1, \dots, \dot{a}_{n-2}) + \wp_2(\dot{a}_2 \dot{b}_{n-1}, \dot{b}_0, \dot{b}_1, \dots, \dot{b}_{n-2}) \\ + \wp_3(\dot{a}_3 \dot{c}_{n-1}, \dot{c}_0, \dot{c}_1, \dots, \dot{c}_{n-2}) + \wp_4(\dot{a}_4 \dot{d}_{n-1}, \dot{d}_0, \dot{d}_1, \dots, \dot{d}_{n-2}) \\ + \wp_5(\dot{a}_5 \dot{e}_{n-1}, \dot{e}_0, \dot{e}_1, \dots, \dot{e}_{n-2}) + \wp_6(\dot{a}_6 \dot{f}_{n-1}, \dot{f}_0, \dot{f}_1, \dots, \dot{f}_{n-2}) \\ + \wp_7(\dot{a}_7 \dot{g}_{n-1}, \dot{g}_0, \dot{g}_1, \dots, \dot{g}_{n-2}) + \wp_8(\dot{a}_8 \dot{h}_{n-1}, \dot{h}_0, \dot{h}_1, \dots, \dot{h}_{n-2}).$$

$$\text{Therefore, } \Upsilon_{(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)} \\ = \Upsilon_{\dot{a}_1}(a) + \Upsilon_{\dot{a}_2}(b) + \Upsilon_{\dot{a}_3}(c) + \Upsilon_{\dot{a}_4}(d) + \Upsilon_{\dot{a}_5}(e) + \Upsilon_{\dot{a}_6}(f) + \Upsilon_{\dot{a}_7}(g) + \Upsilon_{\dot{a}_8}(h)$$

Hence, C_i 's are \dot{a}_i -constacyclic code of length n over \mathcal{F}_p respectively, where $i \in [1, 8]_{\mathbb{Z}}$.

Conversely, let C_i 's are \dot{a}_i -constacyclic codes of length n over \mathcal{F}_p respectively, where $i \in [1, 8]_{\mathbb{Z}}$. Therefore,

$$\wp_1(\dot{a}_1 \dot{a}_{n-1}, \dot{a}_0, \dot{a}_1, \dots, \dot{a}_{n-2}) + \wp_2(\dot{a}_2 \dot{b}_{n-1}, \dot{b}_0, \dot{b}_1, \dots, \dot{b}_{n-2}) \\ + \wp_3(\dot{a}_3 \dot{c}_{n-1}, \dot{c}_0, \dot{c}_1, \dots, \dot{c}_{n-2}) + \wp_4(\dot{a}_4 \dot{d}_{n-1}, \dot{d}_0, \dot{d}_1, \dots, \dot{d}_{n-2}) \\ + \wp_5(\dot{a}_5 \dot{e}_{n-1}, \dot{e}_0, \dot{e}_1, \dots, \dot{e}_{n-2}) + \wp_6(\dot{a}_6 \dot{f}_{n-1}, \dot{f}_0, \dot{f}_1, \dots, \dot{f}_{n-2}) \\ + \wp_7(\dot{a}_7 \dot{g}_{n-1}, \dot{g}_0, \dot{g}_1, \dots, \dot{g}_{n-2}) + \wp_8(\dot{a}_8 \dot{h}_{n-1}, \dot{h}_0, \dot{h}_1, \dots, \dot{h}_{n-2}) \\ = ((\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)\gamma_{n-1}, \gamma_0, \dots, \gamma_{n-2}) \\ = \Upsilon_{(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)}(\gamma).$$

Hence C is a $(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)$ -constacyclic code of length n over \mathcal{R} .

Theorem 7. Let C be a $(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)$ -constacyclic code of length n over \mathcal{R} . Then its dual $C^\perp = \bigoplus_{i=1}^8 \wp_i C_i^\perp$ is a $(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)^{-1}$ -constacyclic code of length n iff C_i 's are \dot{a}_i^{-1} -constacyclic codes.

Proof. From proposition 5 and theorem 6, we can obtained the required result.

Theorem 8. C is a $(1 - 2\mu)$ -constacyclic code over \mathcal{R} iff C_1, C_3, C_4, C_6 are cyclic and C_2, C_5, C_7, C_8 are negacyclic codes.

$$\text{Proof. } \dot{a} = (\dot{a}_0, \dot{a}_1, \dots, \dot{a}_{n-1}) \in C_1, \quad \dot{b} = (\dot{b}_0, \dot{b}_1, \dots, \dot{b}_{n-1}) \in C_2, \\ \dot{c} = (\dot{c}_0, \dot{c}_1, \dots, \dot{c}_{n-1}) \in C_3, \quad \dot{d} = (\dot{d}_0, \dot{d}_1, \dots, \dot{d}_{n-1}) \in C_4, \\ \dot{e} = (\dot{e}_0, \dot{e}_1, \dots, \dot{e}_{n-1}) \in C_5, \quad \dot{f} = (\dot{f}_0, \dot{f}_1, \dots, \dot{f}_{n-1}) \in C_6, \\ \dot{g} = (\dot{g}_0, \dot{g}_1, \dots, \dot{g}_{n-1}) \in C_7, \quad \dot{h} = (\dot{h}_0, \dot{h}_1, \dots, \dot{h}_{n-1}) \in C_8,$$

where $\dot{a}_i, \dot{b}_i, \dot{c}_i, \dot{d}_i, \dot{e}_i, \dot{f}_i, \dot{g}_i, \dot{h}_i \in \mathcal{F}_p$ for $i=0,1,2,\dots,n-1$.

For any arbitrary element $\varphi_i \in \mathcal{R}$ can be uniquely expressed as

$$\gamma_i = \varphi_1 \dot{a}_i + \varphi_2 \dot{b}_i + \varphi_3 \dot{c}_i + \varphi_4 \dot{d}_i + \varphi_5 \dot{e}_i + \varphi_6 \dot{f}_i + \varphi_7 \dot{g}_i + \varphi_8 \dot{h}_i,$$

where $\dot{a}_i, \dot{b}_i, \dot{c}_i, \dot{d}_i, \dot{e}_i, \dot{f}_i, \dot{g}_i, \dot{h}_i \in \mathcal{F}_p$ for $0 \leq i \leq n-1$.

Let $\gamma = (\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_{n-1}) \in C$. First, we assume that C is a $(1-2\mu)$ -constacyclic code of length n over \mathcal{R} , then

$$\begin{aligned} \Upsilon_{(1-2\mu)} &= ((1-2\mu)\gamma_{n-1}, \gamma_0, \dots, \gamma_{n-2}) \in C \\ &= ((1-2\mu)(\varphi_1 \dot{a}_{n-1} + \varphi_2 \dot{b}_{n-1} + \varphi_3 \dot{c}_{n-1} + \varphi_4 \dot{d}_{n-1} + \varphi_5 \dot{e}_{n-1} + \varphi_6 \dot{f}_{n-1} + \varphi_7 \dot{g}_{n-1} + \varphi_8 \dot{h}_{n-1}) \\ &\quad (\varphi_1 \dot{a}_0 + \varphi_2 \dot{b}_0 + \varphi_3 \dot{c}_0 + \varphi_4 \dot{d}_0 + \varphi_5 \dot{e}_0 + \varphi_6 \dot{f}_0 + \varphi_7 \dot{g}_0 + \varphi_8 \dot{h}_0), \\ &\quad \dots, (\varphi_1 \dot{a}_{n-2} + \varphi_2 \dot{b}_{n-2} + \varphi_3 \dot{c}_{n-2} + \varphi_4 \dot{d}_{n-2} + \varphi_5 \dot{e}_{n-2} + \varphi_6 \dot{f}_{n-2} + \varphi_7 \dot{g}_{n-2} + \varphi_8 \dot{h}_{n-2})) \\ &= ((1-\mu-\nu-\varpi+\mu\nu+\nu\varpi+\mu\varpi+\mu\nu\varpi)(\dot{a}_{n-1}, \dot{a}_0, \dot{a}_1, \dots, \dot{a}_{n-2}) \\ &\quad + (\mu-\mu\nu-\mu\varpi+\mu\nu\varpi)(-\dot{b}_{n-1}, \dot{b}_0, \dot{b}_1, \dots, \dot{b}_{n-2}) \\ &\quad + (\nu-\mu\nu-\nu\varpi+\mu\nu\varpi)(\dot{c}_{n-1}, \dot{c}_0, \dot{c}_1, \dots, \dot{c}_{n-2}) \\ &\quad + (\varpi-\mu\varpi-\nu\varpi+\mu\nu\varpi)(\dot{d}_{n-1}, \dot{d}_0, \dot{d}_1, \dots, \dot{d}_{n-2}) \\ &\quad + (\mu\nu-\mu\nu\varpi)(-\dot{e}_{n-1}, \dot{e}_0, \dot{e}_1, \dots, \dot{e}_{n-2}) \\ &\quad + (\nu\varpi-\mu\nu\varpi)(\dot{f}_{n-1}, \dot{f}_0, \dot{f}_1, \dots, \dot{f}_{n-2}) \\ &\quad + (\mu\varpi-\mu\nu\varpi)(-\dot{g}_{n-1}, \dot{g}_0, \dot{g}_1, \dots, \dot{g}_{n-2}) \\ &\quad + \mu\nu\varpi(-\dot{h}_{n-1}, \dot{h}_0, \dot{h}_1, \dots, \dot{h}_{n-2})) \\ &= \varphi_1 F(\dot{a}) + \varphi_2 \Omega(\dot{b}) + \varphi_3 F(\dot{c}) + \varphi_4 F(\dot{d}) + \varphi_5 \Omega(\dot{e}) + \varphi_6 F(\dot{f}) + \varphi_7 \Omega(\dot{g}) + \varphi_8 \Omega(\dot{h}). \end{aligned}$$

Therefore C_1, C_3, C_4, C_6 are cyclic and C_2, C_5, C_7, C_8 are negacyclic codes over the ring \mathcal{F}_p of length n.

Conversely, suppose $\gamma = (\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_{n-1}) \in C$, where

$$\gamma_i = \varphi_1 \dot{a}_i + \varphi_2 \dot{b}_i + \varphi_3 \dot{c}_i + \varphi_4 \dot{d}_i + \varphi_5 \dot{e}_i + \varphi_6 \dot{f}_i + \varphi_7 \dot{g}_i + \varphi_8 \dot{h}_i \text{ and } \dot{a}_i, \dot{b}_i, \dot{c}_i, \dot{d}_i, \dot{e}_i, \dot{f}_i, \dot{g}_i, \dot{h}_i \in \mathcal{F}_p \text{ for } 0 \leq i \leq n-1.$$

If C_1, C_3, C_4, C_6 are cyclic and C_2, C_5, C_7, C_8 are negacyclic codes then

$$F(\dot{a}) \in C_1, \Omega(\dot{b}) \in C_2, F(\dot{c}) \in C_3, F(\dot{d}) \in C_4, \Omega(\dot{e}) \in C_5, F(\dot{f}) \in C_6, \Omega(\dot{g}) \in C_7 \text{ and } \Omega(\dot{h}) \in C_8.$$

Hence, we have

$$\begin{aligned} \Upsilon_{(1-2\mu)} &= ((1-\mu-\nu-\varpi-\mu\nu-\nu\varpi-\mu\varpi+\mu\nu\varpi)F(\dot{a}) \\ &\quad + (\mu-\mu\nu-\mu\varpi+\mu\nu\varpi)\Omega(\dot{b}) + (\nu-\mu\nu-\nu\varpi+\mu\nu\varpi)F(\dot{c}) \\ &\quad + (\varpi-\mu\varpi-\nu\varpi+\mu\nu\varpi)F(\dot{d}) + (\mu\nu-\mu\nu\varpi)\Omega(\dot{e}) \\ &\quad + (\nu\varpi-\mu\nu\varpi)F(\dot{f}) + (\mu\varpi-\mu\nu\varpi)\Omega(\dot{g}) + \mu\nu\varpi\Omega(\dot{h}) \in C \end{aligned}$$

It is given that

$$\begin{aligned} \Upsilon(\gamma) &= \varphi_1 F(\dot{a}) + \varphi_2 \Omega(\dot{b}) + \varphi_3 F(\dot{c}) + \varphi_4 F(\dot{d}) + \varphi_5 \Omega(\dot{e}) + \varphi_6 F(\dot{f}) + \varphi_7 \Omega(\dot{g}) + \varphi_8 \Omega(\dot{h}) \\ &\implies \Upsilon(\gamma) \in C \end{aligned}$$

Therefore, C is a $(1-2\mu)$ -constacyclic code over \mathcal{R} .

Table 1: Behaviour of C_i depending on $(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)$

Units	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
$1 - 2\mu$	\mathcal{C}	\mathcal{N}	\mathcal{C}	\mathcal{C}	\mathcal{N}	\mathcal{C}	\mathcal{N}	\mathcal{N}
$1 - 2\nu$	\mathcal{C}	\mathcal{C}	\mathcal{N}	\mathcal{C}	\mathcal{N}	\mathcal{N}	\mathcal{C}	\mathcal{N}
$1 - 2\varpi$	\mathcal{C}	\mathcal{C}	\mathcal{C}	\mathcal{N}	\mathcal{C}	\mathcal{N}	\mathcal{N}	\mathcal{N}
$-1 + 2\mu$	\mathcal{N}	\mathcal{C}	\mathcal{N}	\mathcal{N}	\mathcal{C}	\mathcal{N}	\mathcal{C}	\mathcal{C}
$-1 + 2\nu$	\mathcal{N}	\mathcal{N}	\mathcal{C}	\mathcal{N}	\mathcal{C}	\mathcal{C}	\mathcal{N}	\mathcal{N}
$-1 + 2\varpi$	\mathcal{N}	\mathcal{N}	\mathcal{N}	\mathcal{C}	\mathcal{N}	\mathcal{C}	\mathcal{C}	\mathcal{N}
$1 - 2\mu + 2\mu\nu + 2\mu\varpi - 2\mu\nu\varpi$	\mathcal{C}	\mathcal{N}	\mathcal{C}	\mathcal{C}	\mathcal{C}	\mathcal{C}	\mathcal{C}	\mathcal{C}
$1 - 2\nu + 2\mu\nu + 2\nu\varpi - 2\mu\nu\varpi$	\mathcal{C}	\mathcal{C}	\mathcal{N}	\mathcal{C}	\mathcal{C}	\mathcal{C}	\mathcal{C}	\mathcal{C}
$1 - 2\varpi + 2\mu\varpi + 2\nu\varpi - 2\mu\nu\varpi$	\mathcal{C}	\mathcal{C}	\mathcal{C}	\mathcal{N}	\mathcal{C}	\mathcal{C}	\mathcal{C}	\mathcal{C}
$-1 + 2\mu - 2\mu\varpi - 2\mu\nu + 2\mu\nu\varpi$	\mathcal{N}	\mathcal{C}	\mathcal{N}	\mathcal{N}	\mathcal{N}	\mathcal{N}	\mathcal{N}	\mathcal{N}
$-1 + 2\nu - 2\mu\nu - 2\nu\varpi + 2\mu\nu\varpi$	\mathcal{N}	\mathcal{N}	\mathcal{C}	\mathcal{N}	\mathcal{N}	\mathcal{N}	\mathcal{N}	\mathcal{N}
$-1 + 2\varpi - 2\mu\varpi - 2\nu\varpi + 2\mu\nu\varpi$	\mathcal{N}	\mathcal{N}	\mathcal{N}	\mathcal{C}	\mathcal{N}	\mathcal{N}	\mathcal{N}	\mathcal{N}

Note: \mathcal{C} and \mathcal{N} stands for cyclic and negacyclic code respectively.

Theorem 9. Let $C = \bigoplus_{i=1}^8 \wp_i C_i$ be a $(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)$ -constacyclic code. Then $C = \langle \wp_1\eta_1, \wp_2\eta_2, \wp_3\eta_3, \wp_4\eta_4, \wp_5\eta_5, \wp_6\eta_6, \wp_7\eta_7, \wp_8\eta_8 \rangle = \mathcal{H}$ and $|C| = p^{8n - \sum_{i=1}^8 \deg(\eta_i)}$, where η_i are generator polynomials of C_i ; $1 \leq i \leq 8$ resp.

Proof. Let C be a $(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)$ -constacyclic code over \mathcal{R} . Then theorem 6 yields

$C_i = \langle \eta_i(C) \rangle \subseteq \mathcal{F}_p[x] / \langle x^n - \alpha_i \rangle$, where $1 \leq i \leq 8$. Also as $C = \bigoplus_{i=1}^8 \wp_i C_i$. So, C is written as $C = \{ \langle \eta(x) \rangle \mid \eta(x) = \sum_{i=1}^8 \wp_i \eta_i, \text{ where } \eta_i(x) \in C_i \text{ for } 1 \leq i \leq 8 \}$

$C \subseteq \mathcal{H} \subseteq \mathcal{R}[x] / \langle x^n - (\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi) \rangle$.

Let us, on the other hand, $\sum_{i=1}^8 \wp_i \eta_i m_i(x) \in \mathcal{H}$,

where $m_i(x) \in \mathcal{R}[x] / \langle x^n - (\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi) \rangle$

then there exist $q_i(x) \in \mathcal{F}_p[x] / \langle x^n - \alpha_i \rangle$ such that $\wp_i m_i(x) = \wp_i q_i(x)$ for $1 \leq i \leq 8$.

Therefore, $\mathcal{H} \subseteq C$. Hence, $C = \mathcal{H}$.

Since $|\psi(C)| = \prod_{i=1}^8 |C_i|$ and $|\psi(C)| = |C|$. So, $|C| = \prod_{i=1}^8 |C_i|$ and

$$|C| = p^k = p^{8n - \sum_{i=1}^8 \deg(\eta_i)}.$$

Theorem 10. Let $C = \bigoplus_{i=1}^8 \wp_i C_i$ be a $(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)$ -constacyclic code over \mathcal{R} . Then

(i) $C^\perp = \langle \sum_{i=1}^8 \wp_i \beta_i^*(x) \rangle$, where $|C^\perp| = p^{\sum_{i=1}^8 \deg(\eta_i)}$ and $1 \leq i \leq 8$.

(ii) $C^\perp = \langle \beta^*(x) \rangle$, where $\beta^*(x) = \sum_{i=1}^8 \wp_i \beta_i^*(x)$ and $\beta_i^*(x)$; $1 \leq i \leq 8$ are the reciprocal polynomial of $\beta_i(x)$ respectively such that $\beta_i \eta_i = x^n - \alpha_i$; $1 \leq i \leq 8$.

The result that follows an imperative CSS construction for QEC codes are implemented in the creation of quantum codes. Here, $[[n, k, \mathfrak{d}]]_p$ is a quantum code with length n , dimension k and minimum distance \mathfrak{d} over \mathcal{F}_p .

Theorem 11. [7] (**CSS Construction**) *Let C_1 and C_2 be linear codes over $GF(p)$ with parameters $[n, k_1, d_1]_p$ and $[n, k_2, d_2]_p$ respectively, where d_1 represents the minimum distance of code C_1 , and d_2 denotes the minimum distance of code C_2 . Assuming that $C_2^\perp \subseteq C_1$, we define $d = \min\{d_1, d_2\}$. Under these conditions, a QEC code C can be constructed with parameters $[[n, k_1 + k_2 - n, d]]_p$. Moreover, if $C_1^\perp \subseteq C$, then \exists a QEC codes C with parameters $[[n, 2k_1 - n, d_1]]_p$. This construction ensures robustness against errors and enhances the performance of quantum codes.*

Lemma 12. [7] *If C is a q -ary linear cyclic or negacyclic code with generator polynomial $\eta(x)$ then C has its dual code iff $x^n - t \equiv 0 \pmod{\eta\eta^*}$ holds, where η^* is the reciprocal polynomial of η and $t = \pm 1$.*

Theorem 13. *Let $C = \bigoplus_{i=1}^8 \wp_i C_i$ be a $(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)$ -constacyclic code over \mathcal{R} . Then $C^\perp \subseteq C$ iff the following conditions hold: $x^n - \dot{\alpha}_i \equiv 0 \pmod{\eta_i\beta_i^*}$ and $\dot{\alpha}_i = \pm 1$.*

Proof. Suppose $x^n - \dot{\alpha}_i \equiv 0 \pmod{\eta_i\beta_i^*}$, where $i \in [1, 8]_{\mathbb{Z}}$

By using lemma 12, $C_i^\perp \subseteq C_i, i \in [1, 8]_{\mathbb{Z}} \implies \wp_i C_i^\perp \subseteq \wp_i C_i; 1 \leq i \leq 8$.

Therefore $\bigoplus_{i=1}^8 \wp_i C_i^\perp \subseteq \bigoplus_{i=1}^8 \wp_i C_i$. Hence the result holds.

Conversely, given that $C^\perp \subseteq C$ then $\bigoplus_{i=1}^8 \wp_i C_i^\perp \subseteq \bigoplus_{i=1}^8 \wp_i C_i$.

Let C_i 's be the linear codes over \mathcal{F}_p then $\wp_i C_i = C \pmod{\wp_i}$ for $i \in [1, 8]_{\mathbb{Z}}$.

So $C_i^\perp \subseteq C_i; i = 1, 2, \dots, 8$. Thus $x^n - \dot{\alpha}_i \equiv 0 \pmod{\eta_i\beta_i^*}$.

Theorem 14. *Let $C = \bigoplus_{i=1}^8 \wp_i C_i$ be a $(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)$. Suppose $C_i^\perp \subseteq C_i, i \in [1, 8]_{\mathbb{Z}}$ then $C^\perp \subseteq C$ exists. Additionally, there is a QEC codes with parameters $[[8n, 2k - 8n, \mathfrak{d}_L]]_p$.*

Proof. Let $r_1 \in \psi(C^\perp) = \psi(C)^\perp$, then $r_1 \in \psi(C)^\perp$. As, $r_1 \in \psi(C)^\perp, \exists \dot{r}_1 \in C^\perp$ such that $r_1 = \psi(\dot{r}_1)$. Since $C^\perp \subseteq C$, therefore $\dot{r}_1 \in C$. Thus $r_1 = \psi(\dot{r}_1) \in \psi(C)$, which implies $\psi(C)^\perp \subseteq \psi(C)$. Since $\psi(C)$ is linear code with parameters $[8n, k, \mathfrak{d}_H]$ then by CSS construction, \exists a quantum code with parameters $[[8n, 2k - 8n, \mathfrak{d}_L]]_p$.

5. Examples

We present several examples to clarify the main result. Specifically, we demonstrate the construction of quantum codes via $(\alpha_1 + \alpha_2\mu + \alpha_3\nu + \alpha_4\varpi + \alpha_5\mu\nu + \alpha_6\nu\varpi + \alpha_7\mu\varpi + \alpha_8\mu\nu\varpi)$ -constacyclic codes of length n over \mathcal{R} .

Example 1. Let $\mathcal{R} = \mathcal{F}_3[u, v, w] / \langle \mu^2 - \mu, \nu^2 - \nu, \varpi^2 - \varpi, \mu\nu - \nu\mu, \nu\varpi - \varpi\nu, \mu\varpi - \varpi\mu \rangle$

Let C be a $(1 - 2\mu + 2\mu\nu + 2\mu\varpi - 2\mu\nu\varpi)$ -constacyclic code over \mathcal{R} of length 15.

$$x^{15} - 1 = (2 + x)^3(1 + x + x^2 + x^3 + x^4)^3$$

$$x^{15} + 1 = (1 + x)^3(1 + 2x + x^2 + 2x^3 + x^4)^3$$

Let $\eta_i(x) = 1 + x + x^2 + x^3 + x^4$, where $1 \leq i \leq 8, i \neq 2$ and $\eta_2(x) = 1 + 2x + x^2 + 2x^3 + x^4$.

Thus $\eta(x) = \sum_{i=1, i \neq 2}^8 \beta_i(1 + x + x^2 + x^3 + x^4) + \beta_2(1 + 2x + x^2 + 2x^3 + x^4)$ is the generating polynomial of C.

As a result $\eta_i(x)\eta_i^*(x)/(x^{15} - 1)$ for $1 \leq i \leq 8, i \neq 2$ and $\eta_2(x)\eta_2^*(x)/(x^{15} + 1)$ resp.

Then from theorem 13, we conclude that $C^\perp \subseteq C$. Moreover, if $\psi(C)$ is a linear code over \mathcal{F}_3 with parameters $[120, 88, 5]$, then theorem 14 yields a quantum code with parameters $[[120, 56, 5]]_3$.

Example 2. Let $\mathcal{R} = \mathcal{F}_5[u, v, w] / \langle \mu^2 - \mu, \nu^2 - \nu, \varpi^2 - \varpi, \mu\nu - \nu\mu, \nu\varpi - \varpi\nu, \mu\varpi - \varpi\mu \rangle$

$$x^{25} - 1 = (4 + x)^{25}$$

$$x^{25} + 1 = (1 + x)^{25}$$

Let C be a $(1 - 2\nu + 2\mu\nu + 2\mu\varpi - 2\mu\nu\varpi)$ -constacyclic code over \mathcal{R} of length 25.

Let $\eta_i(x) = 4 + x$ for $1 \leq i \leq 8, i \neq 3$ and $\eta_3(x) = 1 + x$. Thus $\eta(x) = \sum_{i=1, i \neq 3}^8 \beta_i(4 + x) + \beta_3(1 + x)$ be the generating polynomial of C. Since $\eta_i(x)\eta_i^*(x)/(x^{25} - 1)$ for

$1 \leq i \leq 8, i \neq 3$ and $\eta_3(x)\eta_3^*(x)/(x^{25} + 1)$ resp. Then by theorem 13, we have $C^\perp \subseteq C$.

Additionally, $\psi(C)$ is linear code over \mathcal{F}_5 and the parameters are $[200, 112, 2]$ then by theorem 14, we get the quantum code having parameter $[[200, 104, 2]]_5$.

Table 2: Some examples of quantum codes with various parameters

n	$\eta_1(x) = \eta_3(x) = \eta_4(x) = \eta_6(x)$	$\eta_2(x) = \eta_5(x) = \eta_7(x) = \eta_8(x)$	$\psi(C)$	$[[n, k, \mathfrak{d}]]_p$
9	$-1 + x$	$1 + x$	$[72, 64, 2]$	$[[72, 56, 2]]_3$
15	$1 + x$	$1 + x + x^2$	$[120, 111, 2]$	$[[120, 102, 2]]_3$
30	$1 + x$	$1 + x^2$	$[240, 228, 2]$	$[[240, 216, 2]]_3$
20	$1 + x$	$2 + x^2$	$[160, 148, 2]$	$[[160, 136, 2]]_5$
28	$1 + x$	$1 + 3x + x^2$	$[196, 184, 2]$	$[[196, 172, 2]]_7$
35	$1 + x + x^2 + x^3 + x^4 + x^5 + x^6$	$1 + 4x + x^2 + 4x^3 + x^4 + 4x^5 + x^6$	$[280, 232, 7]$	$[[280, 184, 7]]_5$
45	$1 + x + x^2$	$1 + 4x + x^2$	$[360, 344, 3]$	$[[360, 328, 3]]_5$
55	$4 + x + x^2 + 4x^3 + 2x^4 + x^5$	$1 + 3x + 4x^2 + 4x^3 + x^4 + x^5$	$[440, 400, 6]$	$[[440, 360, 6]]_5$

6. Conclusion

In present paper, we calculated some units having self-inverse and with the

help of these units, we investigate the quantum codes of the ring $\mathcal{F}_p + \mu\mathcal{F}_p + \nu\mathcal{F}_p + \varpi\mathcal{F}_p + \mu\nu\mathcal{F}_p + \nu\varpi\mathcal{F}_p + \mu\varpi\mathcal{F}_p + \mu\nu\varpi\mathcal{F}_p$ under the condition $\mu^2 = \mu, \nu^2 = \nu, \varpi^2 = \varpi, \mu\nu = \nu\mu, \nu\varpi = \varpi\nu, \mu\varpi = \varpi\mu$. It would be interesting to find some other quantum codes over \mathcal{F}_p by taking another Gray map over the ring $\mathcal{F}_p[\mu, \nu, \varpi] / \langle \mu^2 - \mu, \nu^2 - \nu, \varpi^2 - \varpi, \mu\nu - \nu\mu, \nu\varpi - \varpi\nu, \mu\varpi - \varpi\mu \rangle$.

It would also find out the quantum codes of the same ring by taking different conditions and also calculate the quantum codes by taking different rings.

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